Problem 1. Consider a stock in a binary one-period model with value 100 USD at time $t = 0$. We assume that the value increases by 25 percent in the up-state and decreases with 15 percent in the down-state. The risk free interest rate $r = 5$ percent.

(i) Calculate the risk neutral probability distribution at maturity.

(ii) Calculate the arbitrage free price c of a European call option written on this stock with strike price $K = 75$ USD.

Answer:

(i) The risk neutral probability distribution at maturity is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(ii) The call price $c = 28.6$ USD.

Problem 2. Particles in the cosmic radiation with energies exceeding 10^9 eV (one giga electron-volt) reach the surface of the Earth at a rate of approximately 10,000 particles per square meter per second and may cause damage to electronic circuitry and computers resulting in unexpected data errors.

It is believed that these particles arrive independently and at random, and that a fixed small fraction of them lead to computer errors. We consider the random variable X that counts the number of data errors per year in a certain computer caused by cosmic radiation.

(i) What is the distribution of X in the model described above?

Suppose that the mean $E(X) = 2$.

(ii) What is the probability that four or more incidents of computer failure caused by cosmic radiation occur in the computer under consideration during a one year period?

(iii) What is the distribution of the waiting time between computer failures in the computer under consideration, and what is the mean?

Answer:

(i) The Poisson distribution with parameter $\lambda = 2$.

(ii) The probability $P[X \geq 4] = 1 - P[X \leq 3] = 1 - e^{-2}\frac{19}{3} \sim 0.143$.

(iii) Y_1 is exponentially distributed with parameter $\mu = 2$ and mean $E[Y_1] =$ 0.5 years.

Problem 3. Consider the Ito process

$$X_t = B_t \qquad B_0 = 0.$$

Show that

$$\int_0^t B_s \, dB_s = \frac{1}{2} B_t^2 - \frac{t}{2}\,.$$

Hint: Apply Ito's lemma to the function $g(t, x) = \frac{1}{2}x^2$.

Answer:

$$d\left(\frac{1}{2}B_t^2\right) = dY_t = \frac{1}{2}\,dt + B_t\,dB_t \qquad \text{where} \qquad Y_t = g(t, X_t).$$

Problem 4. Assume that the short rate r_t beginning in $r_0 > 0$ is driven by the stochastic differential equation

$$dr_t = e^{-at}\,dt + \sigma dB_t \tag{1}$$

where B_t is the Brownian motion, and a, σ are positive constants.

(i) Integrate (1) to obtain a formula for r_t.

(ii) Calculate the mean $E[r_t]$ and variance $\mathrm{Var}[r_t]$ for $t \geq 0$.

(iii) Find the limit of $E[r_t]$ for $t \to \infty$.

(iv) Why is this a bad model for the term structure?

Answer:

(i) The expression

$$r_t = r_0 + \frac{1}{a}\left(1 - e^{-at}\right) + \sigma B_t$$

is obtained by a straight-forward calculation.

(ii) The expected value and the variance are calculated to be

$$E[r_t] = r_0 + \frac{1}{a}\left(1 - e^{-at}\right) \qquad \text{and} \qquad \mathrm{Var}[r_t] = \sigma^2 t.$$

(iii) Thus $E[r_t] \to r_0 + \dfrac{1}{a}$ for $t \to \infty$.

(iv) The variance $\mathrm{Var}[r_t] \to \infty$ for $t \to \infty$.